

A NOTE ON TWISTOR INTEGRALS

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1. INTRODUCTION

This paper is a brief introduction to twistor integrals from a mathematical point of view. It was inspired by a paper of Hodges [H] which we studied in a seminar at Cal Tech directed by Matilde Marcoli. The idea is to write the amplitude for a graph with n loops and $2n + 2$ propagators using the geometry of pfaffians for sums of rank 2 alternating matrices. (Hodges considers the case of 1 loop and 4 edges). Why is this of interest to a mathematician? The Feynman amplitude is a *period* in the sense of arithmetic algebraic geometry. In parametric form, the amplitude integral associated to a graph Γ with N edges and n loops has the form

$$(1.1) \quad c(N, n) \int_{\delta} \frac{S_1^{N-2n-2} \Omega}{S_2^{N-2n}}.$$

Here S_1 and S_2 are the first and second Symanzik polynomials [BK], [BEK], [IZ], and $\Omega = \sum \pm A_i dA_1 \wedge \cdots \wedge \widehat{dA_i} \wedge \cdots \wedge dA_N$ is the integration form on \mathbb{P}^{N-1} , the projective space with homogeneous coordinates indexed by edges of Γ . The chain of integration δ is the locus of points on \mathbb{P}^{N-1} where all the $A_i \geq 0$. Note Ω , S_1 , S_2 are homogeneous of degrees N , n , $n + 1$ in the A_i , so the integrand is homogeneous of degree 0 and represents a rational differential form. Finally, $c(N, n)$ is some elementary constant depending only on N and n .

Two special cases suggest themselves. In the *log divergent* case when $N = 2n$, the integrand is simply Ω/S_1^2 . The first Symanzik polynomial depends only on the edge variables A_i , so the result in this case is a constant. (If the graph is non-primitive, i.e. has log divergent subgraphs, the integral will diverge. We do not discuss this case.) Inspired by the conjectures of Broadhurst and Kreimer [BrK], there has been a great deal of work done on the primitive log divergent amplitudes.

The polynomial S_1 itself is the determinant of an $n \times n$ -symmetric matrix with entries linear forms in the A_i . The linear geometry of this determinant throws an interesting light on the motive of the hypersurface $X(\Gamma) : S_1 = 0$. For example, one has a ‘‘Riemann-Kempf’’ style

theorem that the dimension of the null space of the matrix at a point is equal to the multiplicity of the point on $X(\Gamma)$, [P], [K]. Furthermore, the projectivized fibre space $Y(\Gamma)$ of these null lines maps birationally onto $X(\Gamma)$ and in some sense “resolves” the motive. Whereas the motive of $X(\Gamma)$ can be quite subtle, the motive of $Y(\Gamma)$ is quite elementary. In particular, it is mixed Tate [B]. (The Riemann-Kempf theorem refers to the map $\pi : \text{Sym}^{g-1}C \rightarrow \Theta \subset J_{g-1}(C)$ where C is a Riemann surface and Θ is the theta divisor. The dimension of the fibre of π at a point of Θ equals the multiplicity of the divisor Θ at the point.)

The second case is $N = 2n + 2$, e.g. one loop and 4 edges. The amplitude is $\int_{\delta} \Omega/S_2^2$ and is a function of external momenta and masses. The second Symanzik has the form

$$(1.2) \quad S_2 = S_2^0(A, q) - \left(\sum_{i=1}^N m_i^2 A_i \right) S_1(A)$$

Here q denotes the external momenta, and $S_2^0(A, q)$ is homogeneous of degree 2 in q and of degree $n+1$ in the A . Moreover, S_2^0 is a quaternionic pfaffian associated to a quaternionic hermitian matrix, [BK], so in the case of zero masses there is again the possibility of linking the motive to the geometry of a linear map. In this note we go further and show for the case $N = 2n + 2$ that S_2 is itself a pfaffian via the calculus of *twistors*.

To avoid issues with convergence for the usual propagator integral, I assume in what follows that the masses are positive and the propagators are euclidean. Note that in (1.4) the pfaffian can vanish where some of the $a_i = 0$. The issues which arise are analogous to issues of divergence already familiar to physicists. They will not be discussed here.

Theorem 1.1. *Let Γ be a graph with n loops and $2n + 2$ edges as above. We fix masses $m_i > 0$ and external momenta q and consider the amplitude*

$$(1.3) \quad \mathcal{A}(\Gamma, q, m) = \int_{\mathbb{R}^{4n}} \frac{d^{4n}x}{\prod_{i=1}^{2n+2} P_i(x, q, m_i)}$$

where the P_i are euclidean. Then there exist alternating bilinear forms Q_i on \mathbb{R}^{2n+2} where Q_i depends on P_i , $1 \leq i \leq 2n + 2$, and a universal constant $C(n)$ depending only on n such that

$$(1.4) \quad \mathcal{A}(\Gamma, q, m) = C(n) \int_{\delta} \frac{\Omega_{2n+1}}{\text{Pfaffian}(\sum_{i=1}^{2n+2} a_i Q_i)^2}$$

Here $\Omega_{2n+1} = \sum \pm a_i da_1 \wedge \cdots \widehat{da_i} \cdots da_{2n+2}$ and δ is the locus on \mathbb{P}^{2n+1} with coordinate functions a_i where all the $a_i \geq 0$.

By way of analogy, the first Symanzik polynomial is given by

$$(1.5) \quad S_1(\Gamma)(a_1, \dots, a_N) = \det\left(\sum_{e \text{ edge}} a_e M_e\right)$$

where M_e is a rank 1 symmetric $n \times n$ -matrix associated to $(e^\vee)^2$, where $e^\vee : H_1(\Gamma, \mathbb{R}) \rightarrow \mathbb{R}$ is the functional which associates to a loop the coefficient of e in that loop. Thus, the amplitude in the case of n loops and $2n$ edges is given by

$$(1.6) \quad \mathcal{A}(\Gamma) = C'(n) \int_{\delta} \frac{\Omega_{2n-1}}{\det(\sum a_i M_i)^2}$$

where $C'(n)$ is another constant depending only on n .

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2. LINEAR ALGEBRA

Fix $n \geq 1$ and consider a vector space $V = k^{2n+2} = ke_1 \oplus \cdots \oplus ke_{2n+2}$. (Here k is a field of characteristic 0.) We write $O = ke_1 \oplus ke_2$ and $I = ke_3 \oplus \cdots \oplus ke_{2n+2}$, so $V = O \oplus I$. $G(2, V)$ will be the Grassmann of 2-planes in V .

We have

$$(2.1) \quad \text{Hom}_k(O, I) \xrightarrow{\iota} G(2, V) \xrightarrow{j} \mathbb{P}\left(\bigwedge^2 V\right).$$

Here $\iota(\psi) = k(e_1 + \psi(e_1)) \oplus k(e_2 + \psi(e_2))$ and $j(W) = \bigwedge^2 W \hookrightarrow \bigwedge^2 V$.

Write V^* for the dual vector space with dual basis e_i^* . We identify $\bigwedge^2 V^*$ with the dual of $\bigwedge^2 V$ in the evident way, so $\langle e_i^* \wedge e_j^*, e_i \wedge e_j \rangle = 1$. For $\alpha \in \bigwedge^2 V^*$, the assignment

$$(2.2) \quad \psi \mapsto \langle (e_1 + \psi(e_1)) \wedge (e_2 + \psi(e_2)), \alpha \rangle$$

defines a quadratic map $q_\alpha : \text{Hom}(O, I) \rightarrow k$.

Lemma 2.1. *Assume $0 \neq \alpha = v \wedge w$ with $v, w \in V^*$. Then the quadratic map q_α has rank 4.*

Proof. It suffices to show $\langle (\sum x_i e_i) \wedge (\sum y_j e_j), v \wedge w \rangle$, viewed as a quadric in the x_i and y_j variables, has rank 4. By assumption v, w are

linearly independent. We can change coordinates so $v = \varepsilon_i^*, w = \varepsilon_j^*$, and $\sum x_i e_i = \sum x'_i \varepsilon_i, \sum y_j e_j = \sum y'_j \varepsilon_j$. The polynomial is then

$$(2.3) \quad \langle (\sum x'_i \varepsilon_i) \wedge (\sum y'_j \varepsilon_j), \varepsilon_i^* \wedge \varepsilon_j^* \rangle = x'_i y'_j - x'_j y'_i.$$

This is a quadratic form of rank 4. \square

Returning to the notation in (2.1), we can write $I = \bigoplus_{i=1}^n I_i$ with $I_i = k e_{2i+1} \oplus k e_{2i+2}$. We can think of $\text{Hom}(O, I) = \bigoplus \text{Hom}(O, I_i)$ as the decomposition of momentum space into a direct sum of Minkowski spaces. We identify $\text{Hom}(O, I_i)$ with the space of 2×2 -matrices, and the propagator with the determinant. With these coordinates, an element in $\text{Hom}(O, I)$ can be written as a direct sum $A_1 \oplus \cdots \oplus A_n$ of 2×2 -matrices. The propagators have the form $\det(a_1 A_1 + \cdots + a_n A_n)$ with $a_i \in k$. The map $\psi : O \rightarrow I$ given by $\psi(e_1) = x_3 e_3 + \cdots + x_{2n+2} e_{2n+2}$ and $\psi(e_2) = y_3 e_3 + \cdots + y_{2n+2} e_{2n+2}$ corresponds to the matrices

$$(2.4) \quad A_i = \begin{pmatrix} x_{2i+1} & x_{2i+2} \\ y_{2i+1} & y_{2i+2} \end{pmatrix}.$$

Lemma 2.2. *Let A_i be as in (2.4). Let*

$$\alpha = \left(\sum_{i=1}^n a_i e_{2i+1}^* \right) \wedge \left(\sum_{i=1}^n a_i e_{2i+2}^* \right) \in \bigwedge^2 V^*.$$

Then the quadratic map q_α in lemma 2.1 is given by

$$(2.5) \quad q_\alpha(A_1 \oplus \cdots \oplus A_n) = \det(a_1 A_1 + \cdots + a_n A_n).$$

Proof. This amounts to the identity

$$(2.6) \quad \det \left(\begin{pmatrix} \sum a_i x_{2i+1} & \sum a_i x_{2i+2} \\ \sum a_i y_{2i+1} & \sum a_i y_{2i+2} \end{pmatrix} \right) = \langle (\sum_{i \geq 3} x_i e_i) \wedge (\sum_{i \geq 3} y_i e_i), (\sum_{i=1}^n a_i e_{2i+1}^*) \wedge (\sum_{i=1}^n a_i e_{2i+2}^*) \rangle.$$

For $i = j$ (resp. $i \neq j$) the coefficient of $a_i a_j$ in this expression is

$$(2.7) \quad x_{2i+1} y_{2i+2} - x_{2i+2} y_{2i+1}$$

$$(2.8) \quad \text{resp. } x_{2i+1} y_{2j+2} - x_{2i+2} y_{2j+1} + x_{2j+1} y_{2j+2} - x_{2j+2} y_{2i+1}.$$

\square

The full inhomogeneous propagator, which in physics notation would be written $(p_1, \dots, p_n) \mapsto (\sum a_i p_i + s)^2$ with the p_i and s 4-vectors,

becomes in the twistor setup

$$\begin{aligned}
(2.9) \quad & \langle (e_1 + \sum_{i \geq 3} x_i e_i) \wedge (e_2 + \sum_{i \geq 3} y_i e_i), \\
& (c_1 e_1^* + c_2 e_2^* + \sum_{i \geq 1} a_i e_{2i+1}^*) \wedge (d_1 e_1^* + d_2 e_2^* + \sum_{i \geq 1} a_i e_{2i+2}^*) \rangle = \\
& \det \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} + c_1 \sum a_i y_{2i+2} - c_2 \sum a_i x_{2i+2} - d_1 \sum a_i y_{2i+1} + \\
& d_2 \sum a_i x_{2i+1} + \det \begin{pmatrix} \sum a_i x_{2i+1} & \sum a_i x_{2i+2} \\ \sum a_i y_{2i+1} & \sum a_i y_{2i+2} \end{pmatrix} = \\
& \det \begin{pmatrix} \sum a_i x_{2i+1} + c_1 & \sum a_i x_{2i+2} + d_1 \\ \sum a_i y_{2i+1} + c_2 & \sum a_i y_{2i+2} + d_2 \end{pmatrix}.
\end{aligned}$$

Remark 2.3. In (2.9), our $\alpha \in \bigwedge^2 V^*$ is of rank 2, i.e. it is decomposable as a tensor and corresponds to an element in $G(2, V) \subset \mathbb{P}(\bigwedge^2 V^*)$, (2.1). If we want to add mass to our propagator, we simply replace α by $\alpha + m^2 e_1^* \wedge e_2^*$, yielding $(\sum a_i p_i + s)^2 + m^2$. The massive α represents a point in $\mathbb{P}(\bigwedge^2 V^*)$ but not necessarily in $G(2, V^*)$.

3. THE TWISTOR INTEGRAL

In this section we take $k = \mathbb{C}$. Consider the maps

$$(3.1) \quad V \times V - S \xrightarrow{\rho} G(2, V) \xrightarrow{j} \mathbb{P}(\bigwedge^2 V).$$

Here $S = \{(v, w) \mid v \wedge w = 0\}$ and $\rho(v, w) = 2$ -plane spanned by v, w .

Lemma 3.1. $V \times V - S / G(2, V)$ is the principal $GL_2(\mathbb{C})$ -bundle (frame bundle) associated to the rank 2 vector bundle \mathcal{W} on $G(2, V)$ which associates to $g \in G(2, V)$ the corresponding rank 2 subspace of V .

Proof. With notation as in (2.1), let $U = \text{Hom}_{\mathbb{C}}(O, I) \subset G(2, V)$. We have

$$(3.2) \quad \rho^{-1}(U) = \{(z_1, \dots, z_{2n+2}, v_1, \dots, v_{2n+2}) \mid \det \begin{pmatrix} z_1 & z_2 \\ v_1 & v_2 \end{pmatrix} \neq 0\}.$$

We can define a section $s_U : U \rightarrow \rho^{-1}(U)$ by associating to $a : O \rightarrow I$ its graph

$$(3.3) \quad s_U(a) := (1, 0, a_1^1, \dots, a_{2n}^1; 0, 1, a_1^2, \dots, a_{2n}^2).$$

Using this section and the evident action of $GL_2(\mathbb{C})$ on the fibres of ρ , we can identify $\rho^{-1}(U) = GL_2(\mathbb{C}) \times U$. The fibre $\rho^{-1}(u)$ for $w \in U$ is precisely the set of framings $w = \mathbb{C}z \oplus \mathbb{C}v$ as claimed. \square

Lemma 3.2. *The canonical bundle $\omega_{G(2,V)} = \mathcal{O}(-2n-2)$ where $\mathcal{O}(-1)$ is the pullback $j^*\mathcal{O}_{\mathbb{P}(\wedge^2 V)}(-1)$.*

Proof. The tautological sequence on $G(2, V)$ reads

$$(3.4) \quad 0 \rightarrow \mathcal{W} \rightarrow V_{G(2,V)} \rightarrow V_{G(2,V)}/\mathcal{W} \rightarrow 0.$$

Here \mathcal{W} is the rank 2 sheaf with fibre over a point of $G(2, V)$ being the corresponding 2-plane in V . One has

$$(3.5) \quad \Omega_{G(2,V)}^1 = \underline{\text{Hom}}(V_{G(2,V)}/\mathcal{W}, \mathcal{W}) = (V_{G(2,V)}/\mathcal{W})^\vee \otimes \mathcal{W}.$$

By definition of the Plucker embedding j above we have $\mathcal{O}_G(-1) = \wedge^2 \mathcal{W}$. The formula for calculating chern classes of a tensor product yields

$$(3.6) \quad c_1(\Omega_G^1) = c_1((V_{G(2,V)}/\mathcal{W})^\vee)^{\otimes 2} \otimes c_1(\mathcal{W})^{\otimes 2n} = \mathcal{O}_G(-2n-2).$$

□

We now fix a point $a \in \mathbb{P}(\wedge^2 V^*)$. Upto scale, a determines a non-zero alternating bilinear form on V which we denote by $Q : (x, y) \mapsto \sum_{\nu, \mu} x_\nu Q^{\nu\mu} y_\mu$. By restriction we may view $Q \in \Gamma(G(2, V), \mathcal{O}(1))$. By the lemma $\omega_G \otimes \mathcal{O}(2n+2) \cong \mathcal{O}_G$, so upto scale there is a canonical meromorphic form ξ on $G(2, V)$ of top degree $4n$ with exactly a pole of order $2n+2$ along $Q=0$. We write

$$(3.7) \quad \xi = \frac{\Xi}{Q^{2n+2}}; \quad 0 \neq \Xi \in \Gamma(G, \omega_G(2n+2)) = \mathbb{C}.$$

Lemma 3.3. *We have*

$$(3.8) \quad H^i(V \times V - S, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 4n+1, 4n+3, 8n+4 \\ (0) & \text{else} \end{cases}.$$

Proof. We compute the dual groups $H_c^*(V \times V - S, \mathbb{Q})$. Note a complex vector space has compactly supported cohomology only in degree twice the dimension. Also, $H_c^1(V - \{0\}) \cong H_c^0(\{0\}) = \mathbb{Q}$. Let $p : S \rightarrow V$ be projection onto the first factor. The fibre $p^{-1}(v) \cong \mathbb{C}$ for $v \neq 0$ and $p^{-1}(0) = V$. It follows that

$$(3.9) \quad H_c^i(S - \{0\} \times V) \cong H_c^{i-2}(V - \{0\}) = (0); \quad i \neq 3, 4n+6.$$

Now the exact sequence

$$(3.10) \quad H_c^i(S - \{0\} \times V) \rightarrow H_c^i(S, \mathbb{Q}) \rightarrow H_c^i(V, \mathbb{Q})$$

yields $H_c^i(S) = \mathbb{Q}$, $i = 3, 4n+4, 4n+6$ and vanishes otherwise. Thus, $H_c^j(V \times V - S) = \mathbb{Q}$; $j = 4, 4n+5, 4n+7, 8n+8$ and vanishes otherwise. Dualizing, we get the lemma. □

Let $R \subset V \times V$ be the zero locus of the alternating form Q on V defined above. Clearly $S \subset R$.

Lemma 3.4. *Assume the alternating form Q is non-degenerate. Then we have*

$$(3.11) \quad H^i(V \times V - R, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 1, 4n + 3, 4n + 4 \\ (0) & \text{else.} \end{cases}$$

Proof. Again let $p : R \rightarrow V$ be projection onto the first factor. We have $p^{-1}(0) = V$ and $p^{-1}(v) \cong \mathbb{C}^{2n+1}$ for $v \neq 0$. It follows that $H_c^i(R - \{0\} \times V) = (0), i \neq 4n + 3, 8n + 6$. As before, this yields $H_c^i(R) = \mathbb{Q}, i = 4n + 3, 4n + 4, 8n + 6$ and zero else. Hence $H_c^j(V \times V - R) = \mathbb{Q}, j = 4n + 4, 4n + 5, 8n + 7, 8n + 8$ and the lemma follows by duality. \square

Note that in the case $n = 0$, $\dim V = 2$ we have $S = R$ and the two lemmas give the same information, which also describes the cohomology of the fibres of the map ρ . Namely, $H^i(\rho^{-1}(pt)) = \mathbb{Q}, i = 0, 1, 3, 4$ and $H^i = (0)$ otherwise.

The form Q induces a quadratic map on $V \times V$ given by $(v, v') \mapsto vQv'$.

Lemma 3.5. *Choose a basis for V and write dv for the evident holomorphic form of degree $4n + 4$ on $V \times V$. Then $\mu := dv/Q^{2n+2}$ is homogeneous of degree 0 and represents a non-trivial class in $H_{DR}^{4n+4}(V \times V - R)$.*

Proof. $V \times V - R$ is affine, so we can calculate de Rham cohomology using algebraic forms. There is an evident \mathbb{G}_m -action which is trivial on cohomology. Writing a form ν as a sum of eigenforms for this action, we can assume the \mathbb{G}_m -action is trivial on ν , which therefore is written $\nu = Fdv/Q^{2n+2+N}$ for some $N \geq 0$ and $\deg F = 2N$. Since Q is non-degenerate, we can write $F = \sum_i F_i \partial Q / \partial v_i$. Let $(dv)_i$ be the form obtained by contracting dv against $\partial / \partial v_i$. Then

$$(3.12) \quad \nu + d\left(\frac{1}{2n+1+N} \sum F_i (dv)_i / Q^{2n+1+N}\right) = Gdv/Q^{2n+1+N}.$$

where G is homogeneous of degree $2(N-1)$. Continuing in this way, we conclude that ν is cohomologous to a constant times dv/Q^{2n+2} . Since by the lemma $H^{4n+4}(V \times V - R) = \mathbb{Q}$, we conclude that $\mu := dv/Q^{2n+2}$ is not exact. \square

If one keeps track of the Hodge structure, lemma 3.4 can be made more precise. One gets e.g. $H^{4n+4}(V \times V - R, \mathbb{Q}) \cong \mathbb{Q}(-2n-3)$. For a

suitable choice of coordinatizations for the two copies of V and a suitable rational scaling for the chain σ representing a class in $H_{4n+4}(V \times V - R, \mathbb{Q})$ we can write the corresponding period as

$$(3.13) \quad \int_{\sigma} d^{2n+2}z \wedge d^{2n+2}v / (\sum z_{\mu} v_{\mu})^{2n+2} = (2\pi i)^{2n+3}.$$

Now we make the change of coordinates $v_{\mu} = \sum_p Q_{\mu}^p w_p$ and deduce

$$(3.14) \quad \int_{\sigma} d^{2n+2}z \wedge d^{2n+2}w / (\sum z_{\mu} Q^{\mu p} w_p)^{2n+2} = \frac{(2\pi i)^{2n+3}}{\det Q}.$$

Here Q is alternating in our case, so $\det Q = \text{Pfaffian}(Q)^2$.

The ‘‘Feynman trick’’ in this context is the integral identity

$$(3.15) \quad \frac{1}{\prod_{i=1}^{2n+2} A_i} = (2n+1)! \int_{0^{2n+2}}^{\infty^{2n+2}} \frac{da_1 \cdots da_{2n+2} \delta(1 - \sum a_i)}{(\sum a_i A_i)^{2n+2}}.$$

We apply the Feynman trick with $A_i = \sum_{\mu,p} z_{\mu} Q_i^{\mu p} w_p$ and integrate over σ

$$(3.16) \quad \begin{aligned} & \int_{\sigma} \frac{d^{2n+2}z \wedge d^{2n+2}w}{\prod_{i=1}^{2n+2} (\sum_{\mu,p} z_{\mu} Q_i^{\mu p} w_p)} = \\ & (2n+1)! \int_{\sigma} d^{2n+2}z \wedge d^{2n+2}w \int_{0^{2n+2}}^{\infty^{2n+2}} \frac{da_1 \cdots da_{2n+2} \delta(1 - \sum a_i)}{(\sum a_i (\sum_{\mu,p} z_{\mu} Q_i^{\mu p} w_p))^{2n+2}} \stackrel{?}{=} \\ & (2n+1)! \int_{0^{2n+2}}^{\infty^{2n+2}} da_1 \cdots da_{2n+2} \delta(1 - \sum a_i) \int_{\sigma} \frac{d^{2n+2}z \wedge d^{2n+2}w}{(\sum_{\mu,p} z_{\mu} (\sum a_i Q_i^{\mu p}) w_p)^{2n+2}} = \\ & (2n+1)! (2\pi i)^{2n+3} \int_{0^{2n+2}}^{\infty^{2n+2}} \frac{da_1 \cdots da_{2n+2} \delta(1 - \sum a_i)}{\text{Pfaffian}(\sum a_i Q_i)^2}. \end{aligned}$$

The integral on the right in (3.16) can be rewritten as a projective integral as on the right in (1.4):

$$(3.17) \quad \int_{0^{2n+2}}^{\infty^{2n+2}} \frac{da_1 \cdots da_{2n+2} \delta(1 - \sum a_i)}{\text{Pfaffian}(\sum a_i Q_i)^2} = \int_{\delta} \frac{\Omega_{2n+1}}{\text{Pfaffian}(\sum_{i=1}^{2n+2} a_i Q_i)^2}.$$

4. PROOF OF THEOREM 1.1

To finish the proof of theorem 1.1, we need to understand the chain of integration σ in (3.16). We also need to choose the alternating forms Q_i on the left side of (3.16) so the resulting integral coincides upto a constant with the Feynman integral in the statement of the theorem (1.3).

Put an hermitian metric $\|\cdot\|$ on V . The induced metric on the bundle of 2-planes defines a submanifold $M \subset V \times V - S$ where M is the set of pairs $(z, v) \in V \times V - S$ such that $\|z\| = \|v\| = 1$ and $\langle z, v \rangle = 0$. M is a \mathbb{U}_2 -bundle which is a reduction of structure of the $GL_2(\mathbb{C})$ bundle $V \times V - S$. The inclusion $M \subset V \times V - S$ is a homotopy equivalence. In particular, the fibre

$$(4.1) \quad (R^4 \rho_* \mathbb{Z})_w \cong H^4(M_w) = H^4(\mathbb{U}_2) = \mathbb{Z} \cdot [\mathbb{U}_2].$$

(\mathbb{U}_2 is a compact orientable 4-manifold, so this follows by Poincaré duality.)

For the base, write $G^0 := G(2, V) - \{Q = 0\}$ where $Q \in \bigwedge^2 V^\vee$ is of rank $2n + 2$. G^0 is affine (and hence Stein) of dimension $4n$, so $H^i(G^0, \mathbb{Z}) = (0)$ for $i > 4n$. Let $\rho^0 : V \times V - R \rightarrow G^0$ be the GL_2 principal bundle obtained by restriction from ρ . We are interested in the class in $H^{4n+4}(V \times V - R, \mathbb{Q})$ (cf. lemma 3.4) dual to σ . The grassmann is simply connected, so by (4.1), necessarily $R^4 \rho_* \mathbb{Z} \cong \mathbb{Z}_G$. Since the fibres of ρ have cohomological dimension 4, we have also

$$(4.2) \quad \mathbb{Q} = H^{4n+4}(V \times V - R, \mathbb{Q}) \cong H^{4n}(G^0, R^4 \rho_*^0 \mathbb{Q}) \cong H^{4n}(G^0, \mathbb{Q}).$$

It is not hard to show in fact that $H^{4n}(G^0, \mathbb{Q}) = \mathbb{Q} \cdot c_2(\mathcal{W})^n$ where \mathcal{W} is the tautological rank 2 bundle on $G(2, V)$ as in (3.4). The interesting question is what if anything this class has to do with the topological closure of real Minkowski space in $G(2, V)$ which is classically the chain of integration for the Feynman integral.

Recall we have Γ a graph with no self-loops and no multiple edges. External edges will play no role in our discussion, so assume Γ has none. The chain of integration for the Feynman integral is \mathbb{R}^{4n} where n is the loop number of Γ . This vector space is canonically identified with $H := H_1(\Gamma, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{R}^4$. In particular, an edge $e \in \text{Edge}(\Gamma)$ yields a functional $e^\vee : H_1(\Gamma, \mathbb{R}) \rightarrow \mathbb{R}$ associating to a loop ℓ the coefficient of e in ℓ .

To avoid divergences, the theorem is formulated for euclidean propagators. Let $q : \mathbb{R}^4 \rightarrow \mathbb{R}$ be $q(x_1, \dots, x_4) = x_1^2 + \dots + x_4^2$. The propagators which appear in the denominator of the integral have the form

$$(4.3) \quad H = H_1(\Gamma, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{R}^4 \xrightarrow{e^\vee \otimes id_{\mathbb{R}^4}} \mathbb{R}^4 \xrightarrow{q} \mathbb{R}.$$

We take complex coordinates in $\mathbb{C}^4 = \mathbb{R}^4 \otimes \mathbb{C}$ of the form

$$(4.4) \quad z_1 = x_1 + ix_2, \quad z_2 = ix_3 + x_4, \quad w_1 = ix_3 - x_4, \quad w_2 = x_1 - ix_2;$$

$$(4.5) \quad x_1 = \frac{z_1 + w_2}{2}, \quad x_2 = \frac{z_1 - w_2}{2i}, \quad x_3 = \frac{z_2 + w_1}{2i}, \quad x_4 = \frac{z_2 - w_1}{2}.$$

In these coordinates $q = z_1 w_2 - z_2 w_1$ and the real structure is $\mathbb{R}^4 = \{(z_1, z_2, -\bar{z}_2, \bar{z}_1) \mid z_j \in \mathbb{C}\}$.

Now take real coordinates for $H_1(\Gamma, \mathbb{R})$ and let $(z_1^k, z_2^k, w_1^k, w_2^k)$, $k \geq 1$ be the resulting coordinates on $H_{\mathbb{C}}$. It is then the case that for each edge e there are real constants $\alpha_k = \alpha_k(e) \in \mathbb{R}$ not all zero, and the propagator for e is

$$(4.6) \quad \det \begin{pmatrix} \sum_{k \geq 1} \alpha_k z_1^k & \sum_{k \geq 1} \alpha_k z_2^k \\ -\sum_{k \geq 1} \alpha_k \bar{z}_2^k & \sum_{k \geq 1} \alpha_k \bar{z}_1^k \end{pmatrix} = \left| \sum_k \alpha_k z_1^k \right|^2 + \left| \sum_k \alpha_k z_2^k \right|^2.$$

Since the linear functionals associated to the various edges e span the dual space to $H_1(\Gamma, \mathbb{R})$, we see that a positive linear combination of the propagators is necessarily positive definite on $H_{\mathbb{R}}$ (i.e. > 0 except at 0.) Using the coordinates z_i^k, w_i^k we can identify $H_{\mathbb{C}}$ with an open set in $G = G(2, 2n+2)$; namely the point with coordinates z, w is identified with the 2-plane of row vectors

$$(4.7) \quad \begin{pmatrix} 1 & 0 & z_1^1 & z_2^1 & z_1^2 & z_2^2 & \dots \\ 0 & 1 & w_1^1 & w_2^1 & w_1^2 & w_2^2 & \dots \end{pmatrix}.$$

We throw in two more coordinates z_1^0, z_2^0 (resp. w_1^0, w_2^0) and view the z_j^k (resp. w_j^k) as coordinates of points in $V_{\mathbb{C}} = \mathbb{C}^{2n+2}$. The fact that the set of non-zero matrices of the form $\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$ is a group under multiplication means that the set of non-zero $2 \times (2n+2)$ -matrices

$$(4.8) \quad \begin{pmatrix} z_1^0 & z_2^0 & z_1^1 & z_2^1 & \dots & z_1^n & z_2^n \\ -\bar{z}_2^0 & \bar{z}_1^0 & -\bar{z}_2^1 & \bar{z}_1^1 & \dots & -\bar{z}_2^n & \bar{z}_1^n \end{pmatrix}$$

is closed in G . It is clearly the closure in G of the real Minkowski space whose complex points are given in (4.7). It will be convenient to scale the rows by a positive real scalar and assume $\sum_{j,k} |z_j^k|^2 = 1$, so the resulting locus is compact in $V \times V - R$. We also scale the bottom row by a constant $e^{i\theta}$ of norm 1. The resulting locus

$$(4.9) \quad \sigma := \left\{ \begin{pmatrix} z_1^0 & z_2^0 & z_1^1 & z_2^1 & \dots & z_1^n & z_2^n \\ -e^{i\theta} \bar{z}_2^0 & e^{i\theta} \bar{z}_1^0 & -e^{i\theta} \bar{z}_2^1 & e^{i\theta} \bar{z}_1^1 & \dots & -e^{i\theta} \bar{z}_2^n & e^{i\theta} \bar{z}_1^n \end{pmatrix} \mid \sum_{j,k} |z_j^k|^2 = 1 \right\} \\ \subset V \times V - R$$

is compact and depends on $4n+4$ real parameters.

Let $Q_e \in \bigwedge^2 V^{\vee}$ be the form which associates to (4.7) the determinant

$$\det \begin{pmatrix} \sum_{k \geq 1} \alpha_k(e) z_1^k & \sum_{k \geq 1} \alpha_k(e) z_2^k \\ \sum_{k \geq 1} \alpha_k(e) w_1^k & \sum_{k \geq 1} \alpha_k(e) w_2^k \end{pmatrix}.$$

Let $a_e > 0$ be constants, and let $\tilde{Q} = \sum_e a_e Q_e \in \bigwedge^2 V^\vee$. Finally, let $Q_0 \in \bigwedge^2 V^\vee$ associate to the matrix (4.8) the minor $z_1^0 \bar{z}_1^0 + z_2^0 \bar{z}_2^0$. It is clear that $Q := Q_0 + \tilde{Q}$ doesn't vanish on any non-zero matrix of the form (4.8). We conclude:

Proposition 4.1. *Let $G(\mathbb{R}) \subset G$ be the set of points (4.8). Then with Q as above, we have $G(\mathbb{R}) \subset G^0 = G - \{Q = 0\}$.*

The locus σ , (4.9), projects down to $G(\mathbb{R})$ with fibre the group \mathbb{U}_2 .

Proposition 4.2. *With this choice of σ we have*

$$(4.10) \quad \int_{\sigma} \frac{d^{2n+2}z \wedge d^{2n+2}w}{Q^{2n+2}} \neq 0.$$

Proof. Let $v_j^{k,\vee}$ be the basis of V^\vee which is dual to the coordinate system z_j^k introduced above. Then one checks that Q as described above is associated to an element

$$(4.11) \quad Q = \sum_{k=0}^n b_k v_1^{k,\vee} \wedge v_2^{k,\vee} \in \bigwedge^2 V^\vee; \quad b_k > 0.$$

Applied to the matrix on the right in (4.9),

$$(4.12) \quad Q(\cdots) = e^{i\theta} \sum_{k=0}^n b_k (|z_1^k|^2 + |z_2^k|^2)$$

Computing $d^{2n+2}z \wedge d^{2n+2}w$ on the right hand side of (4.9) yields

$$(4.13) \quad ie^{(2n+2)i\theta} d\theta \wedge \bigwedge_{j \neq k} (d\bar{z}_1^j \wedge d\bar{z}_2^j) \wedge \sum_k \left((\bar{z}_2^k d\bar{z}_1^k - \bar{z}_1^k d\bar{z}_2^k) \wedge \bigwedge_{j \neq k} (d\bar{z}_1^j \wedge d\bar{z}_2^j) \right).$$

The crucial point is that the $e^{i\theta}$ factor in the integrand (4.10) cancels. Rescaling we can reduce to the case where all the $b_k = 1$. Integrating over σ yields a $2\pi i$ from the $id\theta$ and then an integral over the volume form of the $4n+3$ sphere $\sum_{k=0}^n (|z_1^k|^2 + |z_2^k|^2) = 1$. This is non-zero. \square

The proof of theorem 1.1 is now complete. To summarize, given Γ , one uses the change of coordinates (4.4) in order to rewrite the euclidean propagators P_i as determinants of alternating matrices Q_i . One uses the discussion in section 2, particularly formula (2.9) and remark 2.3, to interpret these propagators with external momenta and masses as elements in $\bigwedge^2 V^\vee$, where $V \cong \mathbb{C}^{\text{Edge}(\Gamma)} \cong \mathbb{C}^{2n+2}$. Using (4.6), one sees that a positive linear combination of the Q_i does not vanish on the locus σ defined in (4.9). This means that the integrand

on the right in (3.16) has poles only on the boundary of the chain of integration where some of the $a_i = 0$. The integral on the left, given our definition of σ , is a constant (depending only on n) times the euclidean amplitude integral.

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